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Examples of Pomonoids of Full Transformations of a Poset

Bana Al Subaiei

Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa, Saudi Arabia

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ABSTRACT

In this research, the partially ordered monoid (simple pomonoid) full transformations of a poset $\mathcal{O}(X)$ is studied, and some related properties are examined. We show that when the poset X is not totally ordered, the pomonoid of all decreasing singular self-maps of a poset X (denoted by S^-) and the pomonoid of all increasing singular self-maps of a poset X (denoted by S^+) may not be generally isomorphic. Some specific partial ordered relations are considered, and the cardinalities of S^- and S^+ under these relations are found. The set of fixed, decreasing, and increasing points of mapping α in $\mathcal{O}(X)$ are also investigated.

KEYWORDS

Posets, pomonoids, full transformations

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1. Introduction and Preliminaries

The semigroup of full transformations on a set has been studied extensively, and many research papers have been published on this subject, such as Howie (1966) and Howie (1971). The semigroup order-preserving (monotone) full transformations of a totally ordered set has also been investigated thoroughly, and substantial literature exists on this subject, such as Howie and Schein (1973), Schein (1975), and Kemprasit and Changphas (2000). The semigroup of all singular self-maps of a totally ordered set has also been studied in Gomes and Howie (1987), Gomes and Howie (1992), and Umar (1992a). As such, it is reasonable to present a new approach to the semigroup order-preserving full transformations of a totally ordered set based on order relations that are in increasing and decreasing orders. Therefore, in the 1990s, the study of semigroup order-increasing full transformations of a totally ordered set and the semigroup order-decreasing full transformations of a totally ordered set were investigated, see Umar (1992a), Umar (1992b), and Umar (1996). All of this research studied the order-preserving full transformations of a totally ordered set as a semigroup. Sohail (2010) considered the pomonoid full transformations of a poset in connection with the ordered representation of a pomonoid. The category of pomonoids has been considered more recently by many researchers, such as Gould and Shaheen (2010), Al Subaiei and Renshaw (2016), Ahanger and Shah (2020), and Al Subaiei (2021). This paper aims to study the order-preserving full transformations of a poset as a pomonoid without limiting the order on the poset for the total order relation.

A set X with a partial order relation is known as a poset. A map $f: X \longrightarrow Y$ where X and Y are posets is called a monotone (order-preserving) whenever $x \leq x'$ then $xf \leq x'f$, where $x, x' \in X$ and $xf, x'f$ in Y. Throughout the study, for any map f, f will be written on the right of its argument as xf, and the set of images of f will be denoted by $Img\ f$.

A semigroup (resp. monoid) with a partial order relation is called a posemigroup (resp. pomonoid) whenever the partial order relation is compatible with the binary operation. This indicates the following: consider the posemigroup T and the partial order relation $\leq$, when $t \leq t'$, then $tt'' \leq t't''$ *and* $t''t \leq t''t'$ for all $t'' \in T$. An element t in a semigroup T is an idempotent if it satisfies the condition $t^2 = t$. Readers can refer to Howie (1995) and Kilp *et al.* (2000) for basic information and terminology on semigroups and monoids and Sohail (2010) and Al Subaiei (2014) for posemigroups and pomonoids.

A full transformation of a set X is the set of all maps from X to X and is usually denoted by $\mathcal{T}(X)$. It is well known that $\mathcal{T}(X)$ is a monoid. The order-preserving full transformations of a poset X is the set of all monotone maps from X to X and is usually denoted by $\mathcal{O}(X)$. This set $\mathcal{O}(X)$ is a pomonoid where the binary relation is composition and the partial order relation is a point-wise order (for any $f, g \in \mathcal{O}(X)$, $f \leq g$ whenever $xf \leq xg$ for all $x \in X$). It is clear that the pomonoid $\mathcal{O}(X)$ is a submonoid of $\mathcal{T}(X)$. The pomonoid $\mathcal{O}(X)$ is known also as the pomonoid full transformations of a poset X.

Let X be a finite poset. The subsemigroup of all singular self-maps of X is

$$Sing = \{\alpha \in \mathcal{O}(X) : |Img\alpha| \leq |X| - 1\}.$$

It is clear that this subsemigroup with the point-wise order is a posemigroup. The set of all decreasing singular self-maps of X is

$$S^- = \{\alpha \in Sing : \forall x \in X, x\alpha \leq x\}$$

while the set of all increasing singular self-maps of X is

$$S^+ = \{\alpha \in Sing : \forall x \in X, x\alpha \geq x\}.$$

The set of shifting points of the mapping α in $\mathcal{O}(X)$ is

$$S(\alpha) = \{x \in X : x\alpha \neq x\}$$

and the cardinality of this set is called shift of α, usually denoted by $s(\alpha)$. The defect of α in $\mathcal{O}(X)$ is the cardinality of the set $Z(\alpha) = X \setminus Img\ \alpha$. The set of fixed points of mapping α in $\mathcal{O}(X)$ is defined as:

$$F(\alpha) = \{x \in X : x\alpha = x\}.$$

The cardinality of the set of fixed points of α, $F(\alpha)$, is denoted by $f(\alpha)$.

2. Results

The primary objective of this work is to study the pomonoid full transformations of a finite poset X, $\mathcal{O}(X)$. The aim is to examine some known results for the semigroup full transformations of a totally ordered set as in Umar (1992a) on the pomonoid full transformations of a poset, where the order on the poset is any partial order relation. As the analog of most properties in the category of

Corresponding Author: Bana Al Subaiei 00966500816416, banajawid@kfu.edu.sa

monoids has two versions in the category of pomonoids, the first with " $=$ " and the other with " $\leq$ ", this will also apply to the set of fixed points. Therefore, the ordered versions will be as follows:

$$F(\alpha)^{<} = \{x \in X : x\alpha \leq x\}$$

and

$$F(\alpha)^{>} = \{x \in X : x\alpha \geq x\}.$$

The poset $F(\alpha)^{<}$ will be called the set of decreasing fixed points of mapping α, while the poset $F(\alpha)^{>}$ will be called the set of increasing fixed points of mapping α. The following result is straightforward to prove.

2.1. Lemma:

Let $\alpha \in \mathcal{O}(X)$. Then,

1. $F(\alpha) = F(\alpha)^{<} \cap F(\alpha)^{>}$.
2. When $\alpha \in S^{-}$ then $F(\alpha)^{<} = X$.
3. When $\alpha \in S^{+}$ then $F(\alpha)^{>} = X$.

2.2. Theorem:

The set S^{-} and S^{+} are posemigroups.

Proof: It is obvious that S^{-} and S^{+} are subsemigroups of $Sing$. We want to prove that S^{-} and S^{+} are posemigroups; specifically, we want to prove that the partial order relation is compatible with the binary operation. Suppose that $\alpha, \beta \in S^{-}$ and $\alpha \leq \beta$. So, for all $x \in X$, we have $x\alpha \leq x\beta$. Then, for any $\gamma \in S^{-}$, we know from the definition of S^{-} that $\gamma \in \mathrm{Sing}$. Hence, $\gamma \in \mathcal{O}(X)$, and so we get $x\alpha\gamma \leq x\beta\gamma$. Thus, $\alpha\gamma \leq \beta\gamma$. Now, since $x\gamma \in X$ and $x\alpha \leq x\beta$ for all $x \in X$, we get $x\gamma\alpha \leq x\gamma\beta$. Thus, $\gamma\alpha \leq \gamma\beta$. Therefore, S^{-} is a posemigroup. By using a similar process, we can show that S^{+} is also a posemigroup. ■

Clearly, we can obtain the following corollary.

2.3. Corollary:

The set S^{-} and S^{+} are subpomonoids of $Sing$.

It is known from Lemma 1.1 of Umar (1992b) that when the order of X is totally ordered, then S^{-} and S^{+} are isomorphic; however, this is not true for any partial order relation as the following example shows.

2.4. Example:

Let $X = \{a, b, c\}$ be a poset with a partial order relation defined as:

Figure 1: The partial order relation of the poset X

$$\begin{array}{ccc} a & & b \\ \backslash & & / \\ & c & \end{array}$$

Then, $\mathcal{O}(X) = \{\gamma_1 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} a & b & c \\ a & a & a \end{pmatrix}$, $\gamma_3 = \begin{pmatrix} a & b & c \\ a & a & c \end{pmatrix}$, $\gamma_4 = \begin{pmatrix} a & b & c \\ b & b & c \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} a & b & c \\ b & b & b \end{pmatrix}$, $\gamma_6 = \begin{pmatrix} a & b & c \\ b & b & a \end{pmatrix}$, $\gamma_7 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$, $\gamma_8 = \begin{pmatrix} a & b & c \\ c & c & c \end{pmatrix}$, $\gamma_9 = \begin{pmatrix} a & b & c \\ a & c & c \end{pmatrix}$, $\gamma_{10} = \begin{pmatrix} a & b & c \\ b & c & c \end{pmatrix}$, $\gamma_{11} = \begin{pmatrix} a & b & c \\ c & b & c \end{pmatrix}$, $\gamma_{12} = \begin{pmatrix} a & b & c \\ c & a & c \end{pmatrix}\}$.

The subset of decreasing singular self-maps of X is $S^{-} = \{\gamma_8, \gamma_9, \gamma_{11}\}$, and the subset of increasing singular self-maps of X is $S^{+} = \emptyset$. It is clear that these are not isomorphic.

2.5. Lemma:

Let X be a poset. Then,

1. If a_i is not comparable with any elements, then for any $\alpha \in S^{-}, a_i\alpha = a_i$.
2. If a_i is not comparable with any elements, then for any $\beta \in S^{+}, a_i\beta = a_i$.
3. If X has a minimum element a, then for any $\alpha \in S^{-}, a\alpha = a$.
4. If X has a maximum element b, then for any $\beta \in S^{+}, b\beta = b$.

Proof:

1. Suppose that a_i is not comparable with any element and $\alpha \in S^{-}$. From the definition of S^{-}, we have $a_i\alpha \leq a_i$. Since a_i is not comparable with any element and the relation on X is a partial order relation, then $a_i\alpha$ must be equal to a_i.
2. The proof is obtained by using a similar argument to case (1).
3. Suppose that X has a minimum element a. From the definition of S^{-} $a\alpha \leq a$. Since a is a minimum element, then $a \leq a\alpha \leq a$. So, as the relation on X is a partial order relation, then $a\alpha$ must be equal to a.
4. The proof is obtained by using a similar argument to case (3). ■

In the following results, we will concentrate on some particular partial order relations and examine some related properties of the full transformations on the pomonoid full transformations of a poset.

2.6. Theorem:

Let $X = \{a_1, a_2, \dots, a_n, a\}$ be a finite set with a partial order relation $\leq_1$ defined such that

Figure 2: The poset X with $\leq_1$

$$\begin{array}{cccc} a_1 & a_2 & \dots & a_n \\ \backslash & | & & / \\ & & a & \end{array}$$

Then, $S^{+} = \emptyset$.

Proof: Suppose that $\alpha \in S^{+}$. Then, $a_i\alpha \geq a_i$ and $a\alpha \geq a$ where $1 \leq i \leq n$. Since there are no elements greater than a_i, then $a_i\alpha = a_i$. From Lemma 2.5, we get that $a\alpha = a$. Hence, α will be the identity map and $\alpha \notin Sing$. Since this is a contradiction, there is no α in S^{+}. ■

2.7. Theorem:

Let $X = \{a_1, a_2, \dots, a_n, a\}$ be a finite set with a partial order relation $\leq_2$ such that

Figure 3: The poset X with $\leq_2$

$$\begin{array}{cccc} & & b & \\ / & | & & \backslash \\ a_1 & a_2 & \dots & a_n \end{array}$$

Then, $S^{-} = \emptyset$.

The proof has a similar argument to the proof of Theorem 2.6.

2.8. Proposition:

Let X be a finite poset.

1. When $X = \{a_1, a_2, \dots, a_n, a\}$ is a poset with the partial order $\leq_1$, then for any $\alpha \in S^{-}$ $a_i\alpha \in \{a, a_i\}$, for all $i = 1, 2, \dots, n$.
2. When $X = \{a_1, a_2, \dots, a_n, b\}$ is a poset with the partial order $\leq_1$, then for any $\beta \in S^{+}$ $a_i\beta \in \{b, a_i\}$, for all $i = 1, 2, \dots, n$.

Proof:

1. Suppose that $\alpha \in S^{-}$. Hence, $a_i\alpha \leq a_i$ and $a\alpha \leq a$. Therefore, $a_i\alpha \in \{a_i, a\}$.
2. The proof is obtained by using a similar procedure to case (1). ■

2.9. Theorem:

Let X be a finite poset.

1. When $X = \{a_1, a_2, \dots, a_n, a\}$ is a poset with the partial order $\leq_1$,

then $|S^-| = C(n,1) + C(n,2) + \cdots + C(n,n) = \frac{n\,(n+1)}{2}$.

2. When $X = \{a_1, a_2, \dots, a_n, b\}$ is a poset with the partial order $\leq_2$, then $|S^+| = C(n,1) + C(n,2) + \cdots + C(n,n) = \frac{n\,(n+1)}{2}$.

Proof:

1. From Proposition 2.8, we know that for any $\alpha \in S^-$, $a_i\alpha \in \{a_i, a\}$. Also, from Lemma 2.5 (3), we get that $a\alpha = a$. When all the elements of X under α have an image equal to a, then there is only one element in S^- having the form $C(n,n) = 1$. When the $n-1$ elements of the a_i have an image equal to a under α, then there are 2 different elements in S^- with the form $C(n, n-1) = 2$. So by using this sequence, we will end when only one element of the a_i has an image equal to a under α and n different elements in S^- have the form $C(n,1) = n$. Therefore, the total number of elements in S^- is equal to $C(n,1) + C(n,2) + \cdots + C(n,n)$. This formula is equal to the nth triangular number, which has the form $\frac{n\,(n+1)}{2}$.
2. The second statement can be proved by using a similar argument to (1). ■

2.10. Example:

In Example 2.4, $|S^-| = C(2,1) + C(2,2) = 2 + 1 = 3$.

2.11. Theorem:

Let X be a poset.

1. If $X = \{a_1, a_2, \dots, a_n, a\}$ is a poset with the partial order $\leq_1$, then the number of α that satisfies $F(\alpha)^< = F(\alpha)^>$ is n^n.
2. If $X = \{a_1, a_2, \dots, a_n, b\}$ is a poset with the partial order $\leq_2$, then the number of α that satisfies $F(\alpha)^< = F(\alpha)^>$ is n^n.

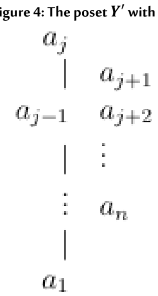
Proof:

1. Suppose that X has the order $\leq_1$ and α satisfies $F(\alpha)^< = F(\alpha)^>$. Since a is the only element that is comparable with all other elements, then $a\alpha = a$. For the other element a_i, where $1 \leq i \leq n$, $a_i\alpha \neq a$, if $a_i\alpha = a$, this means that $a_i \in F(\alpha)^<$ and $a_i \notin F(\alpha)^>$, and this is a contradiction. Hence, each $a_i\alpha \in \{a_1, a_2, \dots, a_n\}$. Moreover, $a_i\alpha$ has n options. Therefore, the number of α that satisfies $F(\alpha)^< = F(\alpha)^>$ is n^n.
2. The second statement can be proved by using a similar argument to (1). ■

Consider the finite poset $Y = \{a_1, a_2, \dots, a_j\}$ with a total order relation defined as $a_1 < a_2 < \cdots < a_j$. Define the map $\rho: Y \to I$ where $I = \{1,2,\dots,j\}$ is a subset of the natural number. Then, it is known that ρ is an order embedding map.

The poset $Y = \{a_1, a_2, \dots, a_j\}$ with a total order relation $a_1 < a_2 < \cdots < a_j$ can be extended to the poset $Y' = \{a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_n\}$ with a partial order relation $\leq_3$ such that $a_1 < a_2 < \cdots < a_j$ where other elements are not comparable with the rest of the elements.

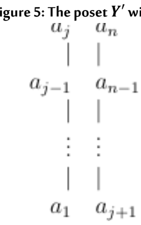
Figure 4: The poset Y' with $\leq_3$



It is clear that the partial order relation $\leq_3$ on Y' is not a total order relation.

Also, the poset $Y = \{a_1, a_2, \dots, a_j\}$ with a total order relation $a_1 < a_2 < \cdots < a_j$ can be extended to the poset $Y' = \{a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_n\}$ with a partial order relation $\leq_4$ such that $a_1 < a_2 < \cdots < a_j$ and $a_{j+1} < a_{j+2} < \cdots < a_n$.

Figure 5: The poset Y' with $\leq_4$



It is obvious that the partial order relation $\leq_4$ on Y' is not a total order relation.

In the following two results, we generalize the result of Lemma 1.1 in Umar (1992b) to the poset Y' when the partial order relation is $\leq_3$ first, and then when the partial order relation is $\leq_4$. The idea of the proof is inspired by Lemma 2.1.1 in Umar (1992a) and Lemma 1.1 in Umar (1992b).

2.12. Theorem:

Let $Y' = \{a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_n\}$ be a poset with the partial ordered relation $\leq_3$. Then, S^- and S^+ are isomorphic subposemigroup of $SingY'$.

Proof: From Theorem 2.2, we know that S^- and S^+ are posemigroups. Now, we want to prove that there exists an order isomorphism map between the two posemigroups S^- and S^+. So, define the map $f: S^- \to S^+$ by $\alpha f = \alpha^*$, where

(i) $a_i\alpha^* = a_i = a_i\alpha$ when $j+1 \leq i \leq n$, and

(ii) $a_i\alpha^* = a_{j-\{a_{(j-i+1)}\alpha\rho\}+1}$ when $1 \leq i \leq j$ where $a_i\rho = i$

where $\alpha \in S^-$ and $\alpha^* \in S^+$.

It is clear that $\alpha^* \in S^+$, since the following statements are satisfied:

(i) when $j+1 \leq i \leq n$, $a_i\alpha^* = a_i$ from Lemma 2.5. Hence, $a_i\alpha^* \geq a_i$, and

(ii) when $1 \leq i \leq j$, we have $a_i\alpha^* = a_{j-\{a_{(j-i+1)}\alpha\rho\}+1} \geq a_{j-\{a_{(j-i+1)}\rho\}+1} = a_{j-(j-i+1)+1} = a_i$. Hence, $a_i\alpha^* \geq a_i$.

First, we will prove that the map f is order embedding. Suppose that $\alpha \leq \beta$. By using the fact that ρ is order embedding, we have the following:

(i) when $j+1 \leq i \leq n$, we have: $\alpha \leq \beta \Leftrightarrow a_i = a_i\alpha \leq a_i\beta = a_i \Leftrightarrow a_i\alpha^* = a_i \leq a_i = a_i\beta^* \Leftrightarrow \alpha f \leq \beta f$.

(ii) when $1 \leq i \leq j$, we have:

$\alpha \leq \beta \Leftrightarrow a_i\alpha \leq a_i\beta \Leftrightarrow a_{(j-i+1)}\alpha \leq a_{(j-i+1)}\beta \Leftrightarrow a_{(j-i+1)}\alpha\rho \leq a_{(j-i+1)}\beta\rho \Leftrightarrow j - \{a_{(j-i+1)}\alpha\rho\} + 1 \leq j - \{a_{(j-i+1)}\beta\rho\} + 1 \Leftrightarrow a_{j-\{a_{(j-i+1)}\alpha\rho\}+1}\rho \leq a_{j-\{a_{(j-i+1)}\beta\rho\}+1}\rho \Leftrightarrow a_{j-\{a_{(j-i+1)}\alpha\rho\}+1} \leq a_{j-\{a_{(j-i+1)}\beta\rho\}+1} \Leftrightarrow f(\alpha) \leq f(\beta)$.

Since f is order embedding, and from Al Subaiei and Renshaw (2016), we see that f is well defined. Now, we want to prove that f is a morphism. To show that, suppose $\alpha f \beta f = \alpha^*\beta^*$. Then, we have the following cases:

(i) when $j+1 \leq i \leq n$, we have $a_i\alpha f\beta f = a_i\alpha^*\beta^* = a_i\beta^* = a_i = a_i(\alpha\beta)^* = a_i(\alpha\beta)f$.

(ii) when $1 \leq i \leq j$, we have $a_i\alpha f\beta f = a_i\alpha^*\beta^* = a_{j-\{a_{(j-i+1)}\alpha\rho\}+1}\beta^* = a_{j-\{a_{(j-[j-\{a_{(j-i+1)}\alpha\rho\}+1]+1)}\beta\rho\}+1} = a_{j-\{a_{\{a_{(j-i+1)}\alpha\rho}\beta\rho\}+1} = a_{j-\{a_{(j-i+1)}\alpha\beta\rho\}+1} = a_i(\alpha\beta)^* = a_i(\alpha\beta)f$. It is clear from the definition of f that f is surjective. Therefore, f is an order isomorphism, and so S^- and S^+ are isomorphic subposemigroups of $\mathrm{Sing}Y'$. ■

2.13. Theorem:

Let $Y' = \{a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_n\}$ be a poset with the partial order relation $\leq_4$. Then, S^- and S^+ are isomorphic subposemigroups of $SingY'$.

The proof has a similar procedure to the proof of Theorem 2.12 above and Lemma 1.1 in Umar (1992b). We simply need to define the map $f: S^- \to S^+$ by $f(\alpha) = \alpha^*$ as the following:

(i) $a_i\alpha^* = a_{n-\{a_{(n-i+1)}\alpha\rho\}+1}$ when $1 \leq i \leq j$.

(ii) $a_i\alpha^* = a_{j-\{a_{(j-i+1)}\alpha\rho\}+1}$ when $j+1 \leq i \leq n$.

where $a_i\rho = i$.

2.14. Proposition:

Let $Y' = \{a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_n\}$ be a poset with the partial order relation $\leq_3$. Then,

1. For any $\alpha \in S^-$, then $a_i \notin Z(\alpha)$ where $j+1 \leq i \leq n$ and $1 \leq$ defect of $\alpha \leq j$.
2. For any $\alpha \in S^+$, then $a_i \notin Z(\alpha)$ where $j+1 \leq i \leq n$ and $1 \leq$ defect of $\alpha \leq j$.

Proof.

1. From Lemma 2.5, we know that $a_i\alpha = a_i$ when $j+1 \leq i \leq n$. Hence, $a_i \in Img\ f$. Therefore, $a_i \notin Z(\alpha)$. Since there are $n-j$ elements not in $Z(\alpha)$, then $1 \leq f(\alpha) \leq j$.
2. The proof of this case has a similar argument to case (1). ■

2.15. Remark:

It is known from Lemma 2.3.1 in Umar (1992a) that $F(\alpha\beta) = F(\alpha) \cap F(\beta)$. However, this is not true when the partial order relation is not totally ordered. In example 2.4, we have $F(\gamma_4) \cap F(\gamma_7) = \{b, c\} \cap \{c\} = \{c\} \neq F(\gamma_4\gamma_7) = F(\gamma_3) = \{a, c\}$. Also, this result does not hold for the ordered version of the set of fixed points, which is the set of decreasing fixed points of α and the set of increasing fixed points of α. Furthermore, in Example 2.4, we have $F(\gamma_9)^< \cap F(\gamma_7)^< = \{a, b, c\} \cap \{c\} = \{c\} \neq F(\gamma_9\gamma_7)^< = F(\gamma_{10})^< = \{b, c\}$ and $F(\gamma_2)^> \cap F(\gamma_6)^> = \{a, c\} \cap \{b, c\} = \{c\} \neq F(\gamma_2\gamma_6)^> = F(\gamma_5)^> = \{b, c\}$. Therefore, in the pomonoid full transformations of a poset, we have the following general cases:

$$F(\alpha\beta) \neq F(\alpha) \cap F(\beta)$$
$$F(\alpha\beta)^< \neq F(\alpha)^< \cap F(\beta)^<$$
$$F(\alpha\beta)^> \neq F(\alpha)^> \cap F(\beta)^>$$

Moreover, it is known from Lemma 2.3.1 in Umar (1992a) that $F(\alpha\beta) = F(\beta\alpha)$. However, this is also not valid when the partial order relation is not totally ordered. In Example 2.4, we have $F(\gamma_4\gamma_7) = F(\gamma_3) = \{a, c\} \neq F(\gamma_7\gamma_4) = F(\gamma_4) = \{b, c\}$, $F(\gamma_9\gamma_7)^< = F(\gamma_{10})^< = \{b, c\} \neq F(\gamma_7\gamma_9)^< = F(\gamma_{12})^< = \{a, c\}$, and $F(\gamma_2\gamma_6)^> = F(\gamma_5)^> = \{b, c\} \neq F(\gamma_6\gamma_2)^< = F(\gamma_2)^< = \{a\}$. Therefore, in the pomonoid full transformations of a poset, we have the following cases in general:

$$F(\alpha\beta) \neq F(\beta\alpha)$$
$$F(\alpha\beta)^< \neq F(\beta\alpha)^<$$
$$F(\alpha\beta)^> \neq F(\beta\alpha)^>$$

2.16. Theorem:

Let $X = \{a_1, a_2, \dots, a_n, a\}$ be finite poset with the partial order relation $\leq_1$. Then, any $\alpha \in S^-$ is an idempotent.

Proof. Suppose that $\alpha \in S^-$. From Lemma 2.5, we know that a $a\alpha = a$. For all a_i, where $1 \leq i \leq n$, it is clear that $a_i\alpha \in \{a_i, a\}$. When $a_i\alpha = a_i$, then $a_i\alpha^2 = a_i$. While when $a_i\alpha = a$ then $a_i\alpha^2 = a$. Therefore, α is an idempotent. ■

By using a similar argument, we can obtain the following:

2.17. Theorem:

Let $X = \{a_1, a_2, \dots, a_n, a\}$ be a finite poset with the partial order relation $\leq_2$. Then, any $\alpha \in S^+$ is an idempotent.

As known from Lemma 2.1.4 in Umar (1992a), α is an idempotent if $f(\alpha) = n - 1$. However, this is not true in general for any partial order relation. Consider the element γ_8 in Example 2.4 γ_8 is an idempotent and $f(\gamma_8) = 1 \neq 2$.

Biography

Bana Al Subaiei

Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa, Saudi Arabia, banajawid@kfu.edu.sa, 00966500816416

Dr. Al Subaiei is a Saudi assistant professor of mathematics and statistics. She has been a vice-dean of Preparatory Year Deanship from 2017 up to the present. She obtained her PhD and master's degrees from the University of Southampton, Southampton, United Kingdom. In addition, she has published some papers in ISI and Scopus journals. She joined various committees in the department, college, and university levels, such as the main committee of scientific chairs and the committee of development and quality assurance at the College of Science. ORCID ID: 0000-0001-6279-4959

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